EVALUATION OF SLOWLY CONVERGENT SERIES.*

By L. D. Ames.

1. General Theory. Any convergent series

$$S = \tfrac{1}{2}u_1 + \tfrac{1}{2}u_2 + u_3 + \cdots \tag{1}$$

can be written in the form:

$$S = \tfrac{1}{2}u_1 + \tfrac{1}{2}[(u_1 + u_2) + (u_2 + u_3) + \cdots]. \tag{2}$$

Repeating the process on the series in brackets we have:

$$S = \tfrac{1}{2}u_1 + \tfrac{1}{4}(u_1 + u_2) + \tfrac{1}{4}[(u_1 + 2u_2 + u_3) + (u_2 + 2u_3 + u_4) + \cdots], \tag{3}$$

$$S = \tfrac{1}{2}u_1 + \tfrac{1}{4}(u_1 + u_2) + \tfrac{1}{8}(u_1 + 2u_2 + u_3)$$
$$+ \tfrac{1}{8}[(u_1 + 3u_2 + 3u_3 + u_4) + (u_2 + 3u_3 + 3u_4 + u_5) + \cdots], \tag{4}$$

and in general:

$$S = \frac{1}{2}u_1 + \frac{1}{2^2}(u_1 + u_2) + \cdots\cdot$$
$$+ \frac{1}{2^k}\left[u_1 + (k-1)u_2 + \frac{(k-1)(k-2)}{1\cdot 2}u_3 + \cdots + u_k\right] + R_k, \tag{5}$$

where

$$R_k = \frac{1}{2^k}\Big[(u_1 + ku_2 + \frac{k(k-1)}{1\cdot 2}u_3 + \cdots\cdot + u_{k+1})$$
$$+ (u_2 + ku_3 + \frac{k(k-1)}{1\cdot 2}u_4 + \cdots + u_{k+2}) + \cdots\Big]. \tag{6}$$

It will be proved in §3 that

$$\lim_{k=\infty} R_k = 0.$$

Assuming this for the present we see that (1) may be written in the form:

$$S = \frac{1}{2}u_1 + \frac{1}{2^2}(u_1 + u_2) + \frac{1}{2^3}(u_1 + 2u_2 + u_3) + \cdots\cdot. \tag{7}$$

---

* Read before the American Mathematical Society at its meeting, April 26, 1902.

It turns out, in a large variety of slowly convergent series in which the positive and negative terms are about evenly balanced in number and aggregate value throughout the series, that series (7) converges rapidly and the error made by stopping at any term is less than the last term added. In practical computation this fact will often be sufficiently evident from a simple inspection of the series. The question of finding an upper limit to the error will be considered later. See §5.

2. Illustrative Examples. We will now illustrate the method described in the last section by applying it to some numerical examples, and in doing so we will indicate a number of slight modifications which are often advantageous.

Example I. $S(x) = \sin x - \dfrac{\sin 2x}{2} + \dfrac{\sin 3x}{3} - \dots$.

To evaluate this series when $x = 50°$ we may tabulate the work as follows:

n	u_n	$u_n + u_{n+1}$	$u_n + 2u_{n+1} + \dots$	$u_n + 3u_{n+1} + \dots$	$u_n + 4u_{n+1} + \dots$	$u_n + 5u_{n+1} + \dots$	$u_n + 6u_{n+1}$
1	+ .7660	+ .2736	− .0521	− .1256	− .0493	+ .0308	+ .0446
2	− .4924	− .3257	− .0735	+ .0763	+ .0801	+ .0138	
3	+ .1667	+ .2522	+ .1498	+ .0038	− .0663		
4	+ .0855	− .1024	− .1460	− .0701			
5	− .1878	− .0436	+ .0759				
6	+ .1443	+ .1195					
7	− .0248						

The first terms of the successive columns correspond to the terms of (7).

$$S = \tfrac{1}{2}\times .7660 + \tfrac{1}{4}\times .2736 - \tfrac{1}{8}\times .0521 + \dots + \tfrac{1}{128}\times .0446 + \dots = .4364 +.$$

But $S(x) = \dfrac{x}{2}$, therefore the correct result is .43633. For $x = 10°$ the same series gives a result correct to four places by the use of four terms.

Example II. $S = \dfrac{1}{\log 2} - \dfrac{1}{\log 3} + \dfrac{1}{\log 4} - \dots$.

We compute the table as before:

n	u_n	$u_n + u_{n+1}$	$u_n + 2u_{n+1} + \cdots$	$u_n + 3u_{n+1} + \cdots$	$u_n + 4u_{n+1} + \cdots$	$u_n + 5u_{n+1} + \cdots$
1	**1.4427**	.5325	.3436	.2547	.2026	.1683
2	**— .9102**	— .1889	— .0889	— .0521	— .0343	
3	**.7213**	**.1000**	**.0368**	**.0178**		
4	— .6216	— .0632	— .0190			
5	.5581	.0442				
6	— .5139					

A slight inspection shows that it is best to sum two terms separately, using the terms in heavy type. The terms above those used need not have been computed.

$$S = 1.4427 - .9102 + \left\{\tfrac{1}{2} \times .7213 + \tfrac{1}{4} \times .1000 + \cdots\right\} = .9239 + .$$

This result is correct to the third place of decimals, as can be proved by §5.

Example III. The reader may compute

$$S(x) = P_0(\mu) - \tfrac{1}{2} P_2(\mu)x + \tfrac{1 \cdot 3}{2 \cdot 4} P_4(\mu)x^2 - \tfrac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} P_6(\mu)x^3 + \cdots,$$

for x equal to or near 1, and μ any desired value. (*Cf.* Byerly, *Fourier's Series and Spherical Harmonics*, p. 152.)

Example IV. $\dfrac{\pi}{4} = 1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \cdots.$

Since the more slowly the original series converges the more rapidly the resulting series converges, we sum ten terms separately and apply the transformation (7) to the next eleven. We obtain $\dfrac{\pi}{4} = .785398163$, correct to nine places (*cf.* §5).

Example V. $S = 1 - \dfrac{1}{2^p} + \dfrac{1}{3^p} - \dfrac{1}{4^p} + \cdots, \quad p = 1.001.$

Sum four terms separately, and apply the transformation to the next seven.

$$S = .693309 + R, \qquad R < .000003. \qquad (\S 5.)$$

Example VI. $S = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdot \cdot \cdot, \quad p = 1.001.$

$$S = \frac{2^p}{2^p - 1}\left(1 - \frac{1}{2^p} + \frac{1}{3^p} \quad \cdot \cdot \cdot \cdot\right) = 1000.563 + R, \qquad \text{(Ex. V.)}$$

$$R < .004.$$

Example VII. $S = 1 + \frac{5 \cdot 7}{8 \cdot 10} + \frac{5 \cdot 7 \cdot 9 \cdot 11}{8 \cdot 10 \cdot 12 \cdot 14} + \cdot \cdot \cdot \cdot .$

$$S = 6\left[1 - \frac{5}{6} + \frac{5 \cdot 7}{6 \cdot 8} - \frac{5 \cdot 7 \cdot 9}{6 \cdot 8 \cdot 10} + \cdot \cdot \cdot\right] = 3.341 + R, \qquad R < .006$$

by the use of four terms (*cf.* §5).

Example VIII. $S = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} \cdot \cdot \cdot \cdot .$

Group the terms thus:

$$S = \left(1 + \frac{1}{3}\right) - \frac{1}{2} + \left(\frac{1}{5} + \frac{1}{7}\right) - \frac{1}{4} + \cdot \cdot \cdot \cdot .$$

Computing the table as in Ex. I, the fifth column is seen to consist of positive terms. It will not be useful to obtain another column. By (5):

$$S = \tfrac{1}{2} \times 1.333 + \tfrac{1}{4} \times .833 + \tfrac{1}{8} \times .676 + \tfrac{1}{16} \times .612 + \tfrac{1}{16}[.593 + .013 + .030$$
$$+ .003 + \cdot \cdot] = 1.037 + \cdot$$

Here the degree of accuracy is not evident. There is strong probability that the result is much closer to the true value than could have been reached by simply adding terms. This example illustrates the limitations of the method; also how it will usually become evident if the method cannot be profitably employed in its entirety, and how it may still be more or less useful in a modified form.

3. Proof of Convergence. We will now complete the proof of (7) by showing that the remainder R_k of (6) approaches zero as k becomes infinite.

$$R_k = R_k' + R_k'' + R_k''',$$

where

$$R'_k = \frac{1}{2^k}\Big[u_1 + (1+k)u_2 + \left(1 + k + \frac{k(k-1)}{1\cdot 2}\right) u_3 + \cdots$$
$$+ \left(1 + k + \frac{k(k-1)}{1\cdot 2} + \cdots + \frac{k(k-1)\cdots(k-r+2)}{1\cdot 2 \ldots (r-1)}\right) u_r \Big].$$

$$R''_k = \frac{1}{2^k}\Big[\left(1 + k + \frac{k(k-1)}{1\cdot 2} + \cdots + \frac{k(k-1)\cdots(k-r+1)}{1\cdot 2 \cdots r}\right) u_{r+1} + \cdots$$
$$+ \left(1 + k + \frac{k(k-1)}{1\cdot 2} + \cdots + \frac{k(k-1)}{1\cdot 2} + k\right) u_k \Big],$$

$$R'''_k = u_{k+1} + u_{k+2} + \cdots.$$

Our theorem will be proved if we can show that a positive ϵ having been chosen at pleasure, a positive m exists such that when $k > m$:

$$|R'_k| < \frac{\epsilon}{3}, \qquad |R''_k| < \frac{\epsilon}{3}, \qquad |R'''_k| < \frac{\epsilon}{3}.$$

Since the u-series converges, m''' can be chosen so that $|R'''_k| < \epsilon/3$ when $k > m'''$.

Similarly choose m'' so that when $k > r > m''$:

$$|u_{r+1} + u_{r+2} + \cdots\cdots u_k| < \epsilon/3.$$

R''_k is of the form:

$$\eta_{r+1}u_{r+1} + \eta_{r+2}u_{r+2} + \cdots\cdots + \eta_k u_k,$$

where

$$0 < \eta_{r+1} < \eta_{r+2} < \cdots\cdots < \eta_k < 1.$$

Hence, by a well-known lemma of Abel's,* when $k > r > m''$,

$$|R''_k| < \epsilon/3.$$

Let M be a positive constant greater than the absolute value of any term of the u-series. Then

$$|R'_k| < \frac{Mr}{2^k}\left[1 + k + \frac{k(k-1)}{1\cdot 2} + \cdots + \frac{k(k-1)\cdots(k-r+2)}{1\cdot 2\cdots(r-1)}\right]$$
$$< \frac{Mr^2}{2^k}\left[\frac{k(k-1)\cdots\cdots(k-r+2)}{1\cdot 2\cdots\cdots(r-1)}\right], \qquad \text{if } k > 2r,$$
$$< \frac{Mr^2}{2^k}\, k^{r-1}.$$

---

* *Cf.* for instance Tannéry: *Théorie des fonctions d'une variable*, p. 95

Put $k = r^2$, then $$|R'_k| < \frac{Mr^{2r}}{2^{r^2}} = M\left(\frac{r^2}{2^r}\right)^r.$$

But $\lim\limits_{r=\infty} \frac{r^2}{2^r} = 0$. $\therefore \lim\limits_{r=\infty} \left(\frac{r^2}{2^r}\right)^r = 0$. $\therefore \lim\limits_{k=\infty} R'_k = 0$.

Therefore m' can be chosen so that $|R'_k| < \epsilon/3$ when $k > m'$.

By taking m greater than the largest of the three quantities m', m'', m''', the truth of the theorem follows at once.

4. A Shorter Method of Computation. By subtracting R_k, in the form given in §3, from S we obtain :

$$S = \frac{1}{2^k}\left[\lambda_1 u_1 + \lambda_2 u_2 + \cdots \lambda_k u_k\right] + R_k, \tag{8}$$

where

$$\lambda_r = 1 + k + \frac{k(k-1)}{1 \cdot 2} + \cdots \frac{k(k-1)\cdots(r+1)}{1 \cdot 2 \cdots (k-r)}. \tag{9}$$

The λ's are the same for all series and may be computed once for all. A few sets are tabulated :

k	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	λ_{10}	λ_{11}
4	15	11	5	1							
7	128	120	99	64	29	8	1				
11	2047	2036	1981	1816	1486	1024	562	232	67	12	1

By the use of this table Ex. II, §2 would be computed as follows :

$$S = 1.4427 - .9102 + \frac{1}{2^4}(15 \times .7213 - 11 \times .6216 + 5 \times .5581 - .5139) + R$$
$$= .9237 + R.$$

Unless we know from other considerations (§5) an upper limit to R the method used in §2 is usually more convenient, since in that method we can observe the rate of convergence.

5. Determination of an Upper Limit for the Error. Let us confine ourselves henceforth to the alternating series :

$$S = v_1 - v_2 + v_3 - v_4 + \cdots, \qquad v_n > 0. \tag{10}$$

Form the successive orders of differences of the absolute value series $v_1 + v_2 + v_3 + \cdots$. If we denote the general term by $\Delta^r v_n$, the series (7) becomes:

$$S = \frac{1}{2} v_1 - \frac{1}{2^2} \Delta^1 v_1 + \frac{1}{2^3} \Delta^2 v_1 - \cdots + (-1)^{r-1} \frac{1}{2^r} \Delta^{r-1} v_1 + \cdots. \tag{11}$$

In some cases the differences can be computed in general terms. Thus let:

$$S = 1 - \frac{a}{b} + \frac{a(a+1)}{b(b+1)} - \frac{a(a+1)(a+2)}{b(b+1)(b+2)} + \cdots, \qquad a < b. \tag{12}$$

Computing the differences and substituting in (11):

$$S = \frac{1}{2} + \frac{1}{2^2} \frac{b-a}{b} + \frac{1}{2^3} \frac{(b-a)(b-a+1)}{b(b+1)} + \cdots. \tag{13}$$

The ratio of any term to the preceding term is less than $\frac{1}{2}$. Hence the error made by stopping at any term is less than the last term added. The series in §2, Exs. IV and VII are of this form.

THEOREM. *If there is a function $f(x)$ which has continuous derivatives of the first $k+1$ orders when $x \geqq 1$; and if the derivatives of even order are positive, those of odd order negative; and $f(n) = v_n = (-1)^{n+1} u_n > 0$; then*

$$|R_k| < \frac{1}{2^k} |u_1|.$$

Proof. Form the sequence

$$f(x_0), \quad f(x_0 + \Delta x), \quad f(x_0 + 2\Delta x), \cdots.$$

It can be proved* that the rth difference quotient $\Delta^r f(x)/\Delta x$ converges uniformly to the value $d^r f(x)/dx^r$ as Δx approaches zero. Hence a positive number δ may be chosen so that for all values of $x \geqq 1$, when $\Delta x < \delta$, $\Delta^r f(x)/\Delta x^r$ and $d^r f(x)/dx^r$ have the same sign. Now choose $\Delta x < \delta$ so that Δx is an aliquot part of unity, *i. e.*, $m \Delta x = 1$. Then the v-series may be formed by taking every mth term of the above sequence, if x_0 is properly chosen. It is easily shown that any difference of the v-series is the sum of positive multiples of dif-

* *Cf.* for instance, Harnack, *Die Elemente der Differential- und Integralrechnung*, §32, where the proof can easily be made to meet the question as to the uniformity of the convergence.

ferences of the same order of this series. Hence $\Delta^r v_n$ is positive when r is even and negative when r is odd. It follows that

$$|\Delta^k v_n| > |\Delta^k v_{n+1}|.$$ But, from (6),

$$R_k = \frac{1}{2^k}(\Delta^k v_1 - \Delta^k v_2 + \Delta^k v_3 \cdot \cdot \cdot \cdot).$$

Hence $$|R_k| < \frac{1}{2^k}|\Delta^k v_1| < \frac{1}{2^k}|u_1|.$$

By the use of this theorem we can prove that the results of Exs. II, IV, V, VI are correct.

6. Application to Divergent Series. If our method be applied to certain divergent series it renders them convergent. This fact becomes of peculiar interest when we apply the method to power series, as we thus get an analytic continuation of the function originally represented. But similar results may be reached by the method of conformal transformation.*

As this treatment of this phase of the subject is simpler, we will not enter upon the discussion here.

HARVARD UNIVERSITY,
CAMBRIDGE, MASSACHUSETTS,
APRIL, 1902.

---

* Painlevé has given a general statement of the method; Paris Thesis, 1887; *Annales de la Faculté des Sciences de Toulouse*, vol. 2 (1888), p. B.1. A number of special developments are worked out in detail by E. Lindelöf, Acta societatis scientiarum Fennicae, vol. 24 (1898), No. 7.